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OPTIMAL STABILIZATION IN THE CRITICAL CASE OF A SINGLE ZERO ROOT†

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The problem of the optimal stabilization [1, 2] of non-linear controlled systems in the critical case of a single zero root [3–5] is considered when the right-hand sides of the equations of the perturbed motion and the integrand in the quality criterion are analytic with respect to the phase coordinates and the control forces. It is assumed that the right-hand side of the critical equation is multiplied by a critical variable and its expansion begins with the terms of the second order. Sufficient conditions for the solvability of the problem are established when the expansion of the integrand in the quality criterion in powers of the phase coordinates and the control forces begin with a positive definite quadratic form, and it is shown that the optimal control is a non-smooth function of the critical variable and has the form of the permissible control proposed in [5] when constructing stabilizing forces in the critical case of a single zero root. An iterative procedure for calculating the optimal control and the optimal Lyapunov function, which is based on results obtained previously [1, 2, 6, 7] and converges for sufficiently small initial perturbations with respect to the non-critical variables, is substantiated. An asymptotic expansion of the optimal result in powers of the critical variable is constructed using perturbation methods [8] and estimates of the accuracy of the approximations are indicated.

1. FORMULATION OF THE PROBLEM

Suppose that the transient dynamics of a controlled system is described by the equations of the perturbed motion

$$\dot{z} = \alpha z^2 + \beta z u + z\mathbf{q}'\mathbf{y} = Z^{(2)}(z,\ \mathbf{y},\ u),\quad \dot{\mathbf{y}} = A\mathbf{y} + \mathbf{b}u + \mathbf{p}z \tag{1.1}$$

where $z \in R$, $\mathbf{y} \in R^n$ are phase coordinates, $u \in R$ is the control force, $A \in R^{nn}$ is a constant matrix, $\mathbf{p}$, $\mathbf{q} \in R$ are constant vectors and a prime denotes transposition.

The transient quality is described by the functional

$$J[u] = \int_0^\infty (\mathbf{y}'R\mathbf{y} + z^2 + u^2)dt \tag{1.2}$$

In order to simplify the calculations and to shorten the length of this paper, it is assumed in (1.1) and (1.2) that the control force u is a scalar quantity and non-linear terms of higher order with respect to the phase coordinates z, y and the control u are omitted, since, subject to the condition that the right-hand side of the critical equation is multiplied by the critical variable and its expansion begins with terms of the second order, the subsequent arguments are not essentially changed and the main features of the problem in question are retained.

Problem 1.1. It is required to find the optimal control $u^0(z, y)$ which stabilizes [1, 2] system (1.1) up to Lyapunov asymptotic stability [3] and minimizes the quality index (1.2)

We shall assume that the following conditions are satisfied.

Condition 1.1. The matrix R is positive-definite.

Condition 1.2. The vectors $\mathbf{b}, A\mathbf{b}, \ldots, A^{n-1}\mathbf{b}$ are linearly independent.

A definition of critical cases [3, 4] of stabilization has been given in [5], the conditions for stabilizability in the critical case of a single zero root are indicated and methods are proposed for constructing stabilizing actions.

Problem 1.1, concerning optimal stabilization [1, 2] in the critical case of a single zero root, is considered in this paper for the first time. This problem possesses a number of special features that

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are uncharacteristic of problems involving the optimal stabilization of non-linear system [2, 6] which makes its solution much more difficult. A procedure for its solution is therefore proposed in the special case when the right-hand side of the critical equation is multiplied by a critical variable. The aim of the investigation was to validate an iterative procedure, which converges in the case of sufficiently small initial perturbations, for calculating the optimal control and the optimal Lyapunov function. In the case of Conditions 1.1 and 1.2, a representation of the optimal control is given in the form of a segment of a divergent series in powers of a critical variable and an asymptotic estimate of the accuracy of the approximation is indicated.

2. OPTIMAL STABILIZATION OF A FIRST-ORDER SYSTEM IN A SPECIAL CASE

We consider auxiliary Problem 1.1 concerning the optimal stabilization of a first-order system when the equations of the perturbed motion have the form

$$\dot{z} = \alpha z^2 + \beta uz + \gamma u^2 = Z^{(2)}(z, u) \tag{2.1}$$

Suppose that the quality index of the control is described by the functional

$$J[u] = \int_0^\infty (z^2 + \rho zu + \lambda u^2)dt \tag{2.2}$$

and that the following conditions are satisfied.

Condition 2.1. As a function of the variable u, the quadratic trinomial $\alpha z^2 + \beta zu + \gamma u^2$ has different real roots, that is, $\beta^2 - 4\alpha\gamma > 0$.

Condition 2.2. The integrand in (2.2) is positive-definite, that is, $\lambda - \rho^2/4 > 0$.

In the case of (2.1), (2.2), the Hamilton–Jacobi equation for Problem 1.1 [2, pp. 484–488] has the form

$$az^2 g + u^0 z(\beta g + \rho) + u^{0^2}(\gamma g + \lambda) + z^2 = 0 \tag{2.3}$$

where $dv/dt = g$.

From the necessary condition for an extremum with respect to u of the left-hand side of (2.3), we calculate the optimal control

$$u^0(z) = -\tfrac{1}{2}(\beta g + \rho)(\gamma g + \lambda)^{-1} z \tag{2.4}$$

Subject to Conditions 2.1 and 2.2, the solution g of Eq. (2.3) in the case of control (2.4) satisfies the inequality

$$\gamma g + \lambda > 0 \tag{2.5}$$

We prove this assertion by contradiction. Initially, suppose that, for a certain $\gamma \neq 0$

$$\gamma g + \lambda = 0 \tag{2.6}$$

Then, the necessary condition for an extremum of the left-hand side of (2.3) takes the form $\beta g + \rho = 0$. Therefore, $g = \lambda\gamma^{-1}$, $\beta = \rho\gamma\lambda^{-1}$. In this case, the basic equation (2.3) is written in the form $(\alpha g + 1)z^2 = 0$. Consequently, $\alpha = \gamma\lambda^{-1} \neq 0$, and Condition 2.1 is written in the form of the inequality

$$(\gamma^2/\lambda^2)(\rho^2 - 4\lambda) > 0$$

which contradicts Condition 2.2. Consequently, Eq. (2.6) is impossible.

It can be verified in a similar manner that the case when Eqs (2.3) and (2.4) have a solution which, for a certain $\gamma \neq 0$, satisfies the inequality $\gamma g + \lambda < 0$, is also impossible. On increasing the positive quantity λ in such a way that this inequality becomes equality (2.6), we now obtain the situation for the transformed functional which has been investigated. Since the integrand in (2.2) remains positive-definite for any increase in the value of λ, we arrive at the conclusion that this case is also impossible. This means that, when Conditions 2.1 and 2.2 are satisfied, Eqs (2.3) and (2.4) have a solution which satisfies inequality (2.5) for any γ.

Hence the control $u^0(z)$ (2.4) is optimal and, in order to calculate it, it is necessary to find the magnitude of g from the equation

$$\alpha g - \tfrac{1}{4}(\beta g + \rho)^2(\gamma g + \lambda)^{-1} + 1 = 0$$

It follows from this that $g \equiv \text{const}$. With regard to the unknown g, we obtain a quadratic equation which, subject to Conditions 2.1 and 2.2, has two different real roots

$$g_{1,2} = -\frac{(2\beta\rho - 4\alpha\lambda - 4\gamma) \mp \sqrt{\Delta}}{2(\beta^2 - 4\alpha\gamma)} \quad (g_1 > 0,\ g_2 < 0) \tag{2.7}$$

$$\Delta = (2\beta\rho - 4\alpha\lambda - 4\gamma)^2 + 16(\beta^2 - 4\alpha\gamma)(\lambda - \rho^2/4)$$

The optimal control (2.4) has the form $u^0(z) = qz$ and the quantity q takes two values

$$q_k = -\tfrac{1}{2}(\beta g_k + \rho)(\gamma g_k + \lambda)^{-1}, \quad k = 1,\ 2 \tag{2.8}$$

According to the construction, it follows from Eq. (2.3) that

$$\operatorname{sign}\frac{dv}{dz} = -\operatorname{sign} Z^{(2)}(z,\ u^0(z))$$

When account is taken of the condition for the asymptotic stability of the trivial solution of the optimal system

$$\dot{z} = Z^{(2)}(z,\ u^0(z)) \tag{2.9}$$

we arrive at the following conclusion.

Theorem 2.1. Suppose that the equation of the perturbed motion has the form of (2.1), the transient quality is estimated by the functional (2.2) and Conditions 2.1 and 2.2 are satisfied. Problem 1.1 then has a unique solution, and the optimal control $u^0(z)$ and the optimal Lyapunov function $V^0(z)$ are defined by the equalities

$$u^0(z) = \begin{cases} q_1 z, & z \geqslant 0 \\ q_2 z, & z \leqslant 0 \end{cases}, \quad V^0(z) = \begin{cases} g_1 z, & z \geqslant 0 \\ g_2 z, & z \leqslant 0 \end{cases}$$

The optimal system of Eq. (2.9) has the form

$$\dot{z} = \alpha z^2, \quad \alpha = \begin{cases} \alpha_1, & z \geqslant 0 \\ \alpha_2, & z \leqslant 0 \end{cases} \tag{2.10}$$

$$\alpha_k = \alpha + q_k\beta + q_k^2\gamma, \quad k = 1,\ 2$$

The coefficients g_1, g_2, q_1, q_2 are calculated using formulae (2.7) and (2.8).

3. OPTIMAL STABILIZATION OF A SYSTEM OF ARBITRARY ORDER

We first consider an auxiliary problem on the optimal stability of the linear system

$$\dot{\mathbf{y}} = A\mathbf{y} + bu$$

when the transient quality is estimated by the functional

$$J[u] = \int_0^\infty (\mathbf{y}'R\mathbf{y} + u^2(\mathbf{y}))dt$$

It is well known [7] that, when Conditions 1.1 and 1.2 are satisfied, this problem has a unique solution, the optimal Lyapunov function is the quadratic form $V^0(\mathbf{y}) = V^{(2)}(\mathbf{y}) = \mathbf{y}'Q\mathbf{y}$ and the optimal control is the linear form $u^0(\mathbf{y}) = u^{(1)}(\mathbf{y}) = \mathbf{c}'\mathbf{y}$, $\mathbf{c} = -Q\mathbf{b}$. Consequently, the optimal system of the first approximation $\dot{\mathbf{y}} = (A + \mathbf{b}\mathbf{c}')\mathbf{y} = P\mathbf{y}$, where $P = A + \mathbf{b}\mathbf{c}'$, is asymptotically stable.

In order to validate the procedure for calculating the solution of Problem 1.1, we find the conditions for the stabilizability of system (1.1). To do this, we calculate the solution $\mathbf{y}^0(z)$ with respect to the variable $\mathbf{y}$ of the following system of algebraic equations

$$P\mathbf{y} + \mathbf{b}v^{(1)}(z) + \mathbf{p}z = 0$$

We have

$$\mathbf{y}^0(z) = -P^{-1}(\mathbf{b}v^{(1)}(z) + \mathbf{p}z)$$

In accordance with Lyapunov's theory [3], we introduce the function

$$Z^{(20)}(z, v^{(1)}(z)) = Z^{(2)}(z, \mathbf{y}^0(z), u^{(1)}(\mathbf{y}^0(z)) + v^{(1)}(z)) = \zeta z^2 + \eta z v^{(1)}(z) \tag{3.1}$$

$$\zeta = \alpha + \beta\mathbf{b}'QP^{-1}\mathbf{p} - \mathbf{q}'P^{-1}\mathbf{p}, \quad \eta = \beta\mathbf{b}'QP^{-1}\mathbf{b} + \beta - \mathbf{q}'P^{-1}\mathbf{b}$$

It is obvious that system (1.1) is stabilized by the equation $u^{(1)}(\mathbf{y}) + v^{(1)}(z)$ if and only if the magnitude of η in (3.1) is non-zero.

The optimal Lyapunov function $V^0(z, \mathbf{y})$ and the optimal control $u^0(z, \mathbf{y})$ which solve Problem 1.1 are sought in the form

$$V^0(z, \mathbf{y}) = \mu z + \mathbf{y}'Q\mathbf{y} + z\mathbf{d}'\mathbf{y} + W(z, \mathbf{y}) \tag{3.2}$$

$$u^0(z, \mathbf{y}) = u^1(\mathbf{y}) + v^{(1)}(z) + u_1(z, \mathbf{y}) \tag{3.3}$$

Here, $W(z, \mathbf{y})$ and $u_1(z, \mathbf{y})$ are terms of a higher order of smallness. Allowing for the fact that the optimal control is defined by the equality

$$u^0(z, \mathbf{y}) = -\tfrac{1}{2}[(\mu\beta + \mathbf{d}'\mathbf{b})z + 2\mathbf{b}'Q\mathbf{y} + \beta z\mathbf{d}'\mathbf{y} + \beta z W_z + \mathbf{b}'\mathbf{W}_\mathbf{y}] \tag{3.4}$$

$$W_z = \partial W(z, \mathbf{y})/\partial z, \quad \mathbf{W}_\mathbf{y} = \partial W(z, \mathbf{y})/\partial \mathbf{y}$$

we obtain the equation

$$\begin{aligned}(\alpha z^2 + z\mathbf{q}'\mathbf{y})(\mu + \mathbf{d}'\mathbf{y} + W_z) + (\mathbf{W}_\mathbf{y})'(A\mathbf{y} + \mathbf{p}z) + (A\mathbf{y} + \mathbf{p}z)'Q\mathbf{y} + \mathbf{y}'Q(A\mathbf{y} + \mathbf{p}z) + \\ + z\mathbf{d}'(A\mathbf{y} + \mathbf{p}z) + \mathbf{y}'R\mathbf{y} + z^2 - \tfrac{1}{4}[(\mu\beta + \mathbf{d}'\mathbf{b})z + 2\mathbf{b}'Q\mathbf{y} + \beta z\mathbf{d}'\mathbf{y} + \beta z W_z + \mathbf{b}'\mathbf{W}_\mathbf{y})]^2 = 0\end{aligned} \tag{3.5}$$

for calculating the function $V^0(z, \mathbf{y})$.

We now select the unknowns μ and d in such a way that the second-order terms in $\{z, \mathbf{y}\}$ vanish in Eq. (3.5). On taking account of the fact that, according to the construction, the matrix Q satisfy the equality

$$A'Q + QA - Q\mathbf{b}\mathbf{b}'Q + R = 0 \tag{3.6}$$

after some algebra we obtain the following equations

$$\mathbf{d} = (P^{-1})'(\mu\beta Q\mathbf{b} - \mu\mathbf{q} - 2Q\mathbf{p}) \tag{3.7}$$

$$\begin{aligned}\mu^2\eta^2 + 4\mu(\mathbf{q}'P^{-1}\mathbf{p} - \eta\mathbf{p}'QP^{-1}\mathbf{b} + \beta\mathbf{p}'QP^{-1}\mathbf{b} - \beta\mathbf{b}'QP^{-1}\mathbf{p} - \alpha) - 4 + 8\mathbf{p}'QP^{-1}\mathbf{p} + \\ + 4(\mathbf{p}'QP^{-1}\mathbf{b})^2 = 0\end{aligned} \tag{3.8}$$

We substitute $A = P + \mathbf{b}\mathbf{b}'Q$ into (3.6) and obtain the matrix equation

$$P'Q + QP + Q\mathbf{b}\mathbf{b}'Q + R = 0$$

We now multiply it on the left by the matrix $QP^{-1}Q^{-1}$ and on the right by $Q^{-1}(P^{-1})'Q$. After some algebra we obtain

$$-1+2\mathbf{p}'QP^{-1}\mathbf{p}+(\mathbf{p}'QP^{-1}\mathbf{b})^2=-1-\mathbf{p}'QP^{-1}Q^{-1}RQ^{-1}(P^{-1})'Q\mathbf{p}<0$$

Consequently, Eq. (3.8) has roots which are opposite in sign.

Theorem 3.1. Suppose that the equations of the perturbed motion have the form of (1.1), the transient quality is estimate by the functional (1.2), Conditions 1.1 and 1.2 are satisfied and the magnitude of η in (3.1) is non-zero. Problem 1.1 then has a unique solution in the first approximation and, moreover, the equation

$$u^{(1)}(z,\mathbf{y})=v^{(1)}(z)+u^{(1)}(\mathbf{y})=-\tfrac{1}{2}[(\mu\beta+\mathbf{d}'\mathbf{b})z+2\mathbf{b}'Q\mathbf{y})] \tag{3.9}$$

stabilizes system 1.1, that is, the system

$$\begin{aligned}\dot{z}&=z^2(\alpha-\tfrac{1}{2}\beta(\mu\beta+\mathbf{d}'\mathbf{b}))+z(\mathbf{q}'-\beta\mathbf{b}'Q)\mathbf{y}\\ \dot{\mathbf{y}}&=P\mathbf{y}+z(\mathbf{p}-\tfrac{1}{2}(\mu\beta+\mathbf{d}'\mathbf{b})\mathbf{b})\end{aligned} \tag{3.10}$$

is asymptotically stable.

Equation (3.8) has two real roots: $\mu_1>0$ and $\mu_1<0$ to which the two values $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$ of the vector $\mathbf{d}$, which is defined by equality (3.7), respectively correspond. In system (3.10), it is necessary to put

$$\mu=\begin{cases}\mu_1, & z>0\\ \mu_2, & z<0\end{cases},\quad \mathbf{d}=\begin{cases}\mathbf{d}^{(1)}, & z>0\\ \mathbf{d}^{(2)}, & z<0\end{cases}$$

The quantity $\nu(\mu)=Z^{(20)}(z,\upsilon^{(1)}(z))/z^2$ is non-zero and is calculated using the equation

$$v^{(1)}(z)=-\tfrac{1}{2}(\mu\beta+\mathbf{d}'\mathbf{b})z$$

and, moreover, $\nu(\mu_1)<0$, $\nu(\mu_2)>0$ and

$$\nu(\mu)=\alpha-\tfrac{1}{2}\beta(\mu\beta+\mathbf{d}'\mathbf{b})+(\mathbf{q}'-\beta b'Q)P^{(-1)}(\mathbf{p}-\tfrac{1}{2}(\mu\beta+\mathbf{d}'\mathbf{b})\mathbf{b})$$

By virtue of (3.10), we denote the complete derivative with respect to time of the function $W(z,\mathbf{y})$ by the symbol $(dW/dt)_{(3.10)}$ and from (3.4)–(3.10), we obtain the equation

$$\begin{aligned}\dot{W}_{(3.10)}&=W_z[z^2(\alpha-\tfrac{1}{2}\beta(\mu\beta+\mathbf{d}'\mathbf{b}))+z(\mathbf{q}'-\beta\mathbf{b}'Q)\mathbf{y}]+\mathbf{W}_y'[P\mathbf{y}+z(\mathbf{p}-\tfrac{1}{2}(\mu\beta+\mathbf{d}'\mathbf{b})\mathbf{b})]=\\ &=\tfrac{1}{4}(\beta z\mathbf{d}'\mathbf{y}+\beta zW_z+\mathbf{b}'\mathbf{W}_y)^2+z^2(\tfrac{1}{2}\beta^2\mu+\tfrac{1}{2}\beta\mathbf{d}'\mathbf{b}-\alpha)\mathbf{d}'\mathbf{y}+z(\beta\mathbf{b}'Q\mathbf{y}-\mathbf{q}'\mathbf{y})\mathbf{d}'\mathbf{y}\end{aligned} \tag{3.11}$$

for calculating the function $W(z,\mathbf{y})$.

An equation of this type arises when solving the problem of the optimal stabilization of non-linear systems [6] and in non-critical situations, when it is possible to prove the analyticity of the optimal Lyapunov function in a certain sufficiently small neighbourhood of the origin of the system of coordinates. However, in the critical case of a single zero root, which is being considered here, the specific feature of the optimal system of the first approximation (3.10) is such that the solution of Eq. (3.11) is, generally speaking, not an analytic function of the critical variable z. This is easily shown by taking the simplest examples where the situation is typical [8, pp. 65–67] when the function $W(z,\mathbf{y})$ can be approximated to any specified accuracy by means of a segment of a divergent series.

On the basis of the properties of the solutions of non-linear system (3.10), it can be shown that the following assertion holds.

Theorem 3.2. Suppose that the conditions of Theorem 3.1 are satisfied. Equation (3.11) then has a continuous unique solution at the point $z=0$ which is defined in the domains $z<0$ and $z>0$. With an accuracy up to z^m in these domains, the function $W(z, y)$ is approximated by a segment of the divergent series

$$W(z,\mathbf{y}) = \sum_{k=1}^{m} [z^k(\alpha_k + W_k(\mathbf{y}))], \quad \alpha_1 = 0 \tag{3.12}$$

where $W_k(\mathbf{y})$ are analytic functions in a sufficiently small neighbourhood of the point $\mathbf{y} = 0$. In the domains $z > 0$ and $z < 0$, the quantities α_k and the functions $W_k(\mathbf{y})$ are uniquely defined for any value of k.

In order to prove Theorem 3.2 we carry out a Lyapunov transformation and change from the variables $\mathbf{y}_i$ to the new variables $\mathbf{x}_i$ in accordance with the equality $\mathbf{x} = \mathbf{y} - \mathbf{y}^0(z)$. We use the notation $W^*(z, \mathbf{x}) = W(z, \mathbf{y}(z)) = W(z, \mathbf{x} + \mathbf{y}^0(z))$ and construct the function $W^*(z, \mathbf{x})$ in the form (3.12), giving the required quantities an asterisk. In this case, we obtain the equation

$$\nu(\mu)\alpha_k^* = \beta_k, \quad (W_x^* V)'\mathbf{x} = W_k^*\mathbf{q}'\mathbf{x} + F_k(\mathbf{x})$$

to calculate the quantities α_k^* and the functions $W_k^*(x)$ when $k = 1, 2, \ldots$.

It can be shown that the quantities β_k and the functions $F_k(\mathbf{x})$ are found from the results of calculations for smaller values of k. Consequently, the validity of the assertions of Theorem 3.2 follows from Lyapunov's theorems [3, pp. 83–100] and the results of Theorem 3.1.

4. EXAMPLE

We will now consider a model example of a problem on the optimal stabilization of a system described by the equations

$$\dot{z} = zy, \quad \dot{y} = u + az \tag{4.1}$$

when the transient quality is described by the functional

$$J[u] = \int_0^\infty (u^2 + z^2 + y^2)dt \tag{4.2}$$

Carrying out the necessary calculations, we obtain

$$V^0(z,y) = \mu z + y^2 + \gamma zy + W(z,y) \tag{4.3}$$

$$\gamma = 2(a \pm \sqrt{a^2+1}), \quad \mu = \pm 2\sqrt{a^2+1}$$

The upper sign corresponds to the domain $z > 0$ and the lower sign to the domain $z < 0$.

The function $W(z, y)$ is the solution of the equation

$$zyW_z - \left(z\frac{\mu}{2} + y\right)W_y = \frac{1}{4}W_y^2 - \gamma zy^2$$

In accordance with (3.13), we seek an approximation of the solution of this equation in the form

$$W(z,y) = zW_1(y) + z^2(\alpha_2 + W_2(y)) + O(z^3)$$

On carrying out the necessary calculations and algebra, we obtain

$$W_1(y) = \gamma\left(\frac{1}{2!}y^2 + \frac{1}{3!}y^3 + O(y^4)\right), \quad \alpha_2 = \frac{\mu}{4}\gamma$$

$$W_2(y) = -\frac{\mu\gamma + \gamma^2}{8}y^2 - \frac{2\mu\gamma + 3\gamma^2}{18}y^3 + O(y^4)$$

It follows from (3.4) and (4.3) that the approximation of the optimal control $u^0(z, y)$ in system (4.1) with the quality index (4.2) is determined by the equality

$$u^0(z,y)=-\frac{1}{2}\left(2y+\left(\gamma+y+\frac{1}{2!}y^2+O(y^3)\right)z+\right.$$

$$\left.+\left(\frac{\mu\gamma}{4}-\frac{\mu\gamma+\gamma^2}{4}y-\frac{2\mu\gamma+3\gamma^2}{6}y^2+O(y^3)\right)z^2+O(z^3)\right)$$

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